"Witelo's use of the mathematical knowledge of his predecessors was to a very great extent uncreative" (p. 28)--the first book of the *Perspectiva* does provide insights into the state of mathematics at the end of the 13th century. For example, Unguru's careful analysis of each proposition and search for possible sources allow him to establish the likely pool of mathematical works available in the Latin West. These include the obvious sources that Witelo frequently quotes--Euclid, Apollonius of Perga, and Alhazen--as well as other sources that can be identified by similarities in arguments--Campanus of Novara's edition of and Theon's additions to the *Elementa,* as well as Eutocius's commentary on *De sphaera et cylindro.* As other likely sources, Unguru lists Pappus' *Mathematicae collectiones,* in unknown translation or in parts; Jordanus's *Geometria;* Theon's recension of Euclid's *Optica* and *Catoptrica;* Theodosius's *Sphaeris,* and Serenus's *De sectione cylindri.* These and other nonmathematical sources came to Witelo through the translations of his friend and colleague at Viterbo, William of Moerbeke.

Unguru's edition of the *Perspectiva* draws principally on three manuscripts and the Risner edition, although eight of a total of twenty-five manuscripts have been consulted and are cited. The translation is careful and stays very close to the Latin text, supplying terms and interpolations only when necessary. An introductory discussion of the life of Witelo, the circumstances surrounding the compilation of the *Perspectiva,* and its significance round out this book. All in all, it is an important addition to the growing collection of textual studies available to persons interested in medieval science and this portion of the history of mathematics.

VON EUDOXOS ZU ARISTOTELES. DAS FORTWIRKEN DER EUDOXISCHEN PROPORTIONENTHEORIE IN DER ARISTOTELISCHEN LEHRE VOM KONTINUUM (Studien zur Antiken Philosophie, Bd. 8). By Hans-Joachim Waschkies. Verlag B. R. Grüner, 1977. 453 pp. Hfl. 90

Reviewed by Ian Mueller
The University of Chicago

The major general claim of this book is that the conscious goals of Aristotle's treatment of the continuous (*to sunechēs*) include the resolution, not only of physical and general philosophical problems, but also of questions in the foundations of mathematics raised by Eudoxus. More specifically, Waschkies attempts to give a developmental account of Aristotle's treatment of the continuous along with a reconstruction of some ideas of Eudoxus, and to argue that at a certain point in Aristotle's development these ideas make themselves felt. The book's primary audience is undoubtedly classical scholars, particularly historians of mathematics and students of Aristotle. The book contains many penetrating analyses of particular passages and concepts which I shall not discuss here. Rather I shall focus on parts of Waschkies' general argument.

Readers familiar with the developmental approach to the study of Aristotle will not be surprised to learn that Waschkies dissects texts into chronological layers and attempts to distinguish the original form of passages from their later reworking. Waschkies is interested only in the early phases of Aristotle's career, roughly the period before Plato's death. He uses two particular assumptions about this period to rule passages out of consideration or treat them as later reworkings: (1) that Aristotle had not yet fully developed the distinction between potentiality and actuality; (2) that in this period Aristotle thought of continuity as applying to spatial objects and not also to time and motion, as he later did. Neither of these assumptions is clearly true, and the second seems to me particularly unlikely, since passages in which spatial objects and time or motion are treated together occur throughout Aristotle's works. The point I wish to stress, however, is that these assumptions lead not only to the elimination or alteration of passages in Aristotle's main discussions of continuity (*Physics*, V,3 and VI,1); they also take these discussions out of the context in which they are preserved for us. For the *Physics* is the work in which Aristotle discusses time and motion, and in which he makes heaviest use of the potentiality-actuality distinction. Nevertheless, Waschkies usually mentions Aristotle's references to these concepts only to dismiss them, and I shall leave the concepts out of account in the sequel.

It is convenient to begin by summarizing the main Aristotelian discussions of continuity. In *Ph.* V,3 Aristotle defines a number of relations between things beginning with "together" (*hama*) and "apart." He then defines contact:

> *Things are in contact* (haptesthai) *if their extremities are together (226b23).*

It is to be noticed that this definition does not presuppose that things in contact are in any particular serial order, but it does presuppose that things in contact have extremities. Aristotle next introduces a notion of order with a formally unsatisfactory definition of "between". He then defines,

> *A thing is in succession* (ephexēs) *if it is after the beginning ... and there is nothing in the same genus between it and what it succeeds, e.g. a line or lines if it is a line, a unit or units if it is a unit, or a house if it is a house (226b34-227a3).*

> *A thing is contiguous* (echomenon) *if it is in succession and in contact [with what it succeeds] (227a6-7).*

> *Continuity is a kind of contiguity. I say that [contiguous] things are continuous when the limit of each at which they are in contact becomes one and the same (227a10-12).*

Aristotle cites his definitions to show that continuity is a subspecies of contact, and simply asserts that contact is a subspecies of succession (227a17-23). It is to be noted that this assertion does not follow from the definitions, since neither contact nor succession is defined in terms of the other; and, insofar as contact does not require an order, one might have things in contact with no natural way of deciding which succeeds which. On the other hand, it does not seem to me implausible to suppose that Aristotle is taking an order for granted and making an intuitively correct but formally unjustified inference. However, Waschkies argues, on the basis of a passage in the *Topics* (IV,2,122b25-28) in which continuity is said to be a species of contact and a passage in Plato's *Parmenides* (148d-149d) in which contact is made a subspecies of succession, that Aristotle is here trying to force a correspondence between his definitions and a standard Academic view.

Aristotle ends *Ph.* V,3 with a curious argument that if points and units are separately existing entities, they are not identical:

> *For contact applies to points, succession to units; and there can be something between points (for every line is between points), but there is no such necessity for units; for there is nothing between one and two (227a29-32).*

I shall discuss part of this argument subsequently. For now I note only that the idea of contact between points seems incompatible with the definition of contact, since points do not have extremities.

In *Ph.* VI,1 Aristotle refers to his previous definitions of continuity, contact, and succession, but recapitulates them differently:

> *Things of which the extremities are one are continuous, things of which the extremities are together are in contact, and things between which there is nothing of the same genus are in succession* (231a22-23).

For Waschkies the variations in content and order are indications that Aristotle is adopting a new perspective. I am inclined to think that Aristotle is simply recalling the intuitive picture he presented in V,3 by saying that continuity is a subspecies of contact, which is a subspecies of succession. In any case Aristotle now draws an important consequence from his definitions:

> *It is impossible for something continuous to be composed of indivisibles, e.g. a line of points, if a line is continuous and a point indivisible. (a) For the limits of points are not one; for there is not one part of an indivisible which is an extremity and another part. (b) Nor are the limits together; for there is no limit of a thing without parts, since a limit and that of which it is a limit are different (231a24-29).*

It is to be noticed that no direct sense can be made of the notion of a continuous line, if one relies on Aristotle's definition of continuity, according to which continuity is a relation between things like lines. We might well think that a continuous thing is one whose components are related by continuity, and Waschkies appears to hold this view. For he takes (b), which is clearly intended to show that indivisibles can't be in contact, as an unexpected addition. However, I am inclined to think that Aristotle calls a line continuous in an intuitive sense, and in VI,1 argues that a line cannot be composed of points which are continuous, in contact, or in succession. The argument against the last possibility comes eight lines after the arguments against the first two:

> *Nor can a point be in succession to a point ... in such a way that a length is composed of points.... For things between which there is nothing of the same genus are in succession, but what is between points is always a line ... (231b6-9).*

Before this argument Aristotle inserts a second argument designed to show that the points on a line can neither be continuous nor in contact. He refers to his first argument for the case of continuity, and then says,

> *All things are in contact as whole with whole or as part with part or as whole with part. But since the indivisible is partless, it is necessarily in contact as whole with whole. But if it is in contact as whole with whole, there will be no continuity. For the continuous has different parts and is thus divided into different and spatially separated parts (231b2-6).*

Here Aristotle allows what is excluded in the preceding argument and in the definitions of *Ph.* V,3, namely, that points might be in contact. He argues that such contact can only be coincidence and hence cannot yield an extended line.

The anomaly of the argument of 231b2-6 is a cornerstone of Waschkies' theorizing. He sees the argument as a foreign body which must have an outside source, namely, Eudoxus. I shall discuss Waschkies' theories shortly. Here I shall remark only that Aristotle, like many people, frequently uses principles he does not accept to refute views he opposes. The anomaly of 231b 2-6 may not be as great as Waschkies supposes.

The final argument in *Ph.* VI,1 against a continuum's being composed of indivisibles is very cumbersome.

> *A. [If a continuum could be composed of indivisibles], it could be divided into indivisibles, if* (eiper) *each thing is divided into that of which it is composed.*
>
> *B. But no continuous thing was divisible into indivisibles.*
>
> *C. Nor can there be anything of another kind between the points...; for if there could be, it is clear that it must be either (a) indivisible or (b) divisible, and if it is divisible, it must be divisible into either (b1) indivisibles or (b2) into things which are always divisible. This is the continuous.*
>
> *D. But it is clear that every continuum is divisible into things which are always divisible.*
>
> *E. For if it were divisible into indivisibles, indivisibles would be in contact with indivisibles; for the limit of continuous things is one and in contact (231b10-18).*

In B, Aristotle states as an established fact what he has not yet shown, but goes on to show in E. E is formulated as an argument for the infinite divisibility of continua (D), but the argument shows that for Aristotle infinite divisibility and nondivisibility into indivisibles are equivalent notions. In E, Aristotle refers to his previous arguments against contact of indivisibles, and eliminates their being continuous on the ground that continuity is a kind of contact. The possibility of a continuum's being divided into successive indivisibles not in contact might seem to be open, but this possibility is apparently closed in C. There Aristotle argues that (a) and (b1) violate principle B, leaving only (b2) as a possibility, the possibility which Aristotle declares to be actual for the continuous.

This kind of odd logical order is not unusual in Aristotle, and, if one makes allowances for a certain degree of imprecision, the reasoning is sound. However, there are two features of the argument which should be made more explicit. The first is Aristotle's assumption in A that a thing is divisible into its components. Aristotle does not admit the possibility that a line might be composed of points but not divisible into them. More important is Aristotle's assertion that a continuum is infinitely divisible (D). For Waschkies this is the assertion that a thing with continuous components is infinitely divisible, but I take it that Aristotle is still using an intuitive undefined notion of a continuum and arguing that it cannot be divided into indivisibles which are either continuous or in contact or in succession.

On either interpretation Aristotle gets into difficulty in *Ph.* VI,2 when he says,

> *I call continuous that which is divisible into things which are always divisible (232b24-25)*,

and infers the continuity of time from its infinite divisibility. For, although Aristotle has established that continuous things are infinitely divisible, he has not proved, and quite clearly could not prove, that infinitely divisible things are continuous. Waschkies interprets this situation as showing that Aristotle is working with two different definitions of continuity,

> *K.1.* a *is continuous if and only if it is infinitely divisible,*
>
> *K.2.* b *is continuous with* c *if and only if they have a common extremity,*

and illegitimately assuming their equivalence. I suggest as an alternative interpretation that Aristotle defines a continuum (wrongly) in terms of K.1 in *Ph.* VI.2, having argued in VI,1, on the basis of his definitions of the relations "continuous" (K.2), "in contact," and "successive," that a continuum (intuitively conceived) is infinitely divisible. I shall not pursue this suggestion here, but turn instead to Waschkies' account of the complicated texts I have been describing.

For Waschkies K.1 and K.2 are reflections of two different traditions taken over by Aristotle. The account of the continuous as infinitely divisible goes back to Anaxagoras' reflections on the paradoxes of Zeno. Waschkies claims that this account underlies the early "Grundschicht" of Aristotle's treatment of the infinite (*Ph.* III.4-8). On the other hand, K.2 and the other definitions of *Ph.* V,3 are for Waschkies Aristotle's early attempt to come to grips in a fairly precise way with the question "When does a multiplicity of parts constitute a single thing?" This philosophical question has its roots in the Eleatic conception of being, and was treated by the natural philosophers Anaxagoras and Democritus. Of particular importance for Waschkies is the claim that in this tradition there was no concern for the inner structure (Feinstruktur) of a continuum, or, roughly speaking, no attempt to be precise about the nature of the parts making up a unit.

Waschkies focuses the question of inner structure in what he calls the composition and division problems: Can a continuum be composed of indivisibles? In what way or ways can a continuum be divided? He argues that Aristotle's concern with inner structure in *Ph.* VI,1 (as opposed to V,3) is an indication of a third influence, the mathematical work of Eudoxus. To describe Waschkies conception of this influence, it is necessary to describe some points from his account of the development of Greek mathematics. According to him, Democritus established the volume equality of pyramids with equal heights and congruent triangular bases by imagining them resting in the same plane and using planes parallel to their bases to establish a one-one correspondence between congruent triangles out of which each pyramid might be said to be composed. He then used elementary geometric arguments to establish that any pyramid has one-third the volume of a prism with the same height and base, and a formally illegitimate exhaustion argument to establish the corresponding result for cones and cylinders.

The famous cone fragment, in which Democritus asks about the equality or inequality of the "surfaces of the segments" made by passing a plane through a cone parallel to its base, shows that Democritus was in some sense aware of the composition problem. However, according to Waschkies, he simply bypassed it, and was able to do so because his treatment of pyramids with congruent bases required establishing a one-one correspondence between the sections made in two pyramids but did not require a summation of the sections. Eudoxus, influenced by Zeno, took up the composition problem. In Waschkies' view, although Zeno himself had asserted that an extended magnitude could not be composed of unextended ones, he did not provide logical argumentation of the kind a mathematician like Eudoxus would require.

Waschkies takes over the standard view according to which Eudoxus provided the satisfactory proofs of Democritus' theorems and related results which we find in book XII of Euclid's *Elements,* and worked out the general theory of proportion which is now book V. Waschkies also accepts what seems to be the standard account of the development of Greek proportion theory: a first stage in which the theory is only applicable to commensurable magnitudes, and a is said to be to b as c to d if and only if the Euclidean algorithm applied to the pairs (a,b) and (c,d) yields the same numerical expression for the ratios; a second, "anthyphairetic" stage in which this is said if and only if the algorithm applied to the same pairs yields the same (possibly infinite) sequence of integers (i.e., the denominators of the expansions of the ratios into continued fractions); and a third, Eudoxian stage in which "a is to b as c is to d" means that for any integers m and n, $m \cdot a$ and $m \cdot c$ are alike greater than, equal to, or less than $n \cdot b$ and $n \cdot d$.

Waschkies devotes some time to arguing that Eudoxus was the first to formulate

> El. *V, def. 4: Magnitudes which when multiplied can exceed one another are said to have a ratio to one another,*

which Waschkies, like most scholars, takes to be an expression of the Archimedean condition. Editors customarily point to two applications of this definition in the *Elements,* one in V,8, a proposition whose proof can almost certainly be ascribed to

Eudoxus, and one in X,1, the assertion that, given two magnitudes, subtracting more than half from the greater, more than half from what remains, and so on, will eventually produce a magnitude smaller than the smaller of the original two magnitudes. This latter proposition is immediately applied in X,2, the assertion that the Euclidean algorithm yields a common measure if there is one, an assertion naturally associated with the anthyphairetic theory of ratio; otherwise, X,1 is applied only in book XII. Waschkies argues that X,1 was tacitly assumed in the original proof of X,2, and found its present position as the result of subsequent editorial work. The editor is said to have borrowed X,1 from Eudoxus, who had placed it at the beginning of what we know as book XII, where it plays a fundamental role.

Waschkies' account of the relation between the work of Democritus and that of Eudoxus suggests that Eudoxus might have been concerned with the inner structure of continua. But nothing in the *Elements* requires this assumption. In particular, V, def. 4 may have no connection with either V,8 or X,1 (and hence none with book XII); the definition can be interpreted as indicating only that incommensurable magnitudes are covered in the theory of Book V [1].

To buttress his claim that V,def. 4, is related to the question of inner structure Waschkies assigns to Eudoxus the following definitions from the *Elements*.

El.I,def. 1: A point is that which has no parts.
El.I,def. 2: A line is breadthless length.
El.I,def. 5: A surface is that which has length and breadth only.
El.XI,def. 1: A solid is that which has length, breadth, and depth.

These definitions are said to form the basis of Eudoxus' argument that an n-dimensional thing cannot be a sum of $(n - 1)$-dimensional things, an argument which we have already seen at *Ph.* VI,1,231b2-6, and which occurs in different forms in the Grundschicht of *On Coming to Be and Passing Away,* I,2, and most explicitly in *On Indivisible Lines.* The possible connection between these definitions and the argument is clearest in the case of the definition of point; for the argument uses the partlessness of points to infer coincidence from contact. However, it is easy enough to see that the same considerations could be used in connection with the definitions of line, surface, and solid to argue against a surface's being composed of lines or a solid of planes.

It is now possible to describe in more detail the signs of Eudoxian influence which Waschkies finds in Aristotle. This influence is already seen in the difficult argument at *Ph.* V,3, 227a29-32, where Aristotle says that points can be in contact. Aristotle also denies here and again in VI,1, 231b6-9 that points can be in succession, on the ground that between distinct points there is or can be a straight line. Normally commentators fill out this argument by saying that the straight line can be bisected to yield an intermediate point, but Waschkies takes the straight line to be the thing of the same genus as the points. Using this argument it is possible to provide a conjectural reconstruction of Eudoxus' refutation of the possibility of a straight line's being composed of points: either the points are in contact and they coincide, or they are not in contact and a straight line can be drawn between them.

Although *Ph*. V,3 contains traces of Eudoxian influence, for Waschkies it stems from a time when Aristotle was not fully cognizant of the problem of inner structure. Contact with Eudoxus made him aware of this problem, and at a subsequent stage, which is clearest in *On Coming to Be and Passing Away*, I,2, he relied heavily on Eudoxus' argument to cope with it. Finally, by the time of *Ph*. VI,1 Aristotle has come to see that he can cope with the problem using his own definition of continuity, although he also includes some unnecessary Eudoxian arguments in the chapter as well.

I turn now to Waschkies' treatment of the division problem, a problem which may be broken down into three questions.

1. Can two lines differ by a point?
2. Can a line be divided into points?
3. Where can a line be divided?

According to Waschkies, the division problem arose in Greek mathematics in connection with question 1. In *The Quadrature of the Parabola* Archimedes says that he assumes as a *lēmma* one similar to that assumed by earlier geometers in the proofs of certain propositions now found in book XII of the *Elements*. The *lēmma* asserts that, given three n-dimensional magnitudes, the difference between two of them can always be multiplied enough times to exceed the third. For Waschkies this assertion amounts to a generalization of the statement that the difference between two lines is not a point. He suggests that Eudoxus did not formulate this assertion as an explicit assumption because he believed that he could prove it.

Waschkies' reconstruction of the proof is based primarily on *Ph*. IV,8, 215b12-20. The proof is based on the assumption that the difference between any two n-dimensional magnitudes bears a ratio to the greater magnitude and hence to any n-dimensional magnitude; by *El*.V,def. 4, this assumption means that if the difference between two lines were a point, a sum of points could exceed a line, and a fortiori such a sum could be equal to a line; but the Eudoxian solution to the composition problem shows this to be impossible. Given V,def. 4, the assumption on which this argument is based clearly begs the question. Moreover, Archimedes' discussion suggests that he considers himself to have uncovered an assumption made tacitly by his predecessors [2]. In the absence of any clear evidence that Eudoxus did formulate the argument, I am inclined to be skeptical about Waschkies' proposal.

Aristotle's principal concern is to give a negative answer to question 2. We have already seen that he does it in *Ph.* VI, 1,231b16-18 by means of the same general strategy which Waschkies ascribes to Eudoxus in connection with question 1: reduction to the impossibility of a magnitude's being composed of indivisibles. Waschkies holds that Aristotle's main purpose in his discussion of the division problem in VI,1 is to meld K.1 and K.2, and so prepare the way for the use of K.1 in VI,2. As we have seen, Aristotle fails to achieve this purpose. At the end of his book Waschkies raises the question whether this failure undermines his view that Aristotle's treatment of continuity is intended to contribute to the study of mathematical foundations. For Waschkies this question reduces to the question whether the mathematicians of Aristotle's time could have recognized the inadequacy for geometric purposes of the condition that lines be infinitely divisible. It is now, of course, a commonplace that infinite divisibility corresponds to density, which does not yield what we call continuity. Waschkies argues persuasively that Aristotle's contemporaries could not have recognized the inadequacy, because they took for granted an intuitive answer to question 3 according to which a geometric magnitude is divisible "everywhere" (*pantēi*); Aristotle thinks K.1 is a geometrically satisfactory account of continuity because he takes the same intuitive answer for granted. Since his mistake would not have been recognized by the mathematicians of his time, Waschkies concludes that it cannot be used as evidence against his own view that Aristotle was engaged in foundational studies.

Provided the notion of foundational studies is taken in a sufficiently general sense, there seems to me every reason to accept this view. For although Waschkies' historical claims about Eudoxus, Aristotle's relation to Eudoxus, and Aristotle's own intellectual development seem tenuous, Aristotle's treament of continuity includes an attempt to analyze the relation between mathematical points and lines; moreover, this analysis is far and away the most sophisticated and precise to survive from antiquity and, to my knowledge, from any time before the nineteenth century. Waschkies' careful investigation makes clear the significance of this analysis. For such careful scholarly work we must be grateful.

NOTES

1. I have discussed these questions in Ian Mueller, 1981, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, Cambridge, Mass. (MIT Press).

2. This point has been argued for convincingly in M. Dehn,

1937, "Beziehungen zwischen der Philosophie und der Grundlagen der Mathematik in Altertum," *Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik,* Abteilung B 4, 1-28. See especially pp. 19-22.

THE HISTORY OF STATISTICS IN THE 17TH AND 18TH CENTURIES AGAINST THE CHANGING BACKGROUND OF INTELLECTUAL, SCIENTIFIC AND RELIGIOUS THOUGHT. Lectures by Karl Pearson given at University College, London, during the academic sessions 1921-1933. Edited by E. S. Pearson. New York, Macmillan, 1978. xix, 744 pp.

Reviewed by James H. Cassedy
National Library of Medicine
Bethesda, Maryland

In 1921, at the age of 64, Karl Pearson began a lecture course at University College, London, on the history of statistics. Over the next dozen years, Pearson developed and greatly expanded these lectures. He did not attempt to cover the entire subject in any one term, but rather each year dealt in depth with certain selected aspects or periods. By 1933, however, when he retired, he had covered in great detail large portions of the history of statistics up through the 19th century.

Upon Pearson's death in 1936 the manuscripts of these lectures were found among his papers. Whether or not Pearson himself had earlier modified the manuscripts is not clear. In any case, during the next few years Pearson's widow did some arranging and editing of the lectures with a view toward their possible publication. However, Major Greenwood, Udny Yule, and other outside readers strongly advised against publication without considerable additional changes and corrections. Now, four decades later, the lectures have been published by Pearson's son as a memorial to his father, somewhat further annotated and edited, though presumably not to the extent suggested by Greenwood and Yule.

The result is a very large volume, one that is both less and more than the manuscript left by Pearson. On the one hand, sizable portions of material have been deleted in the interest of saving space. On the other hand, the editor has arranged the lectures in broad chronological order, has given a short account of their background, has inserted numerous comments of his own at various points in his father's text, and has provided a name index. The volume comes to us, therefore, as a mixed bag, one that is filled with much to admire and instruct, but one that also has its weaknesses.

The idea of having a history of statistics by Karl Pearson is an appealing one. It is instructive to see how the biographer